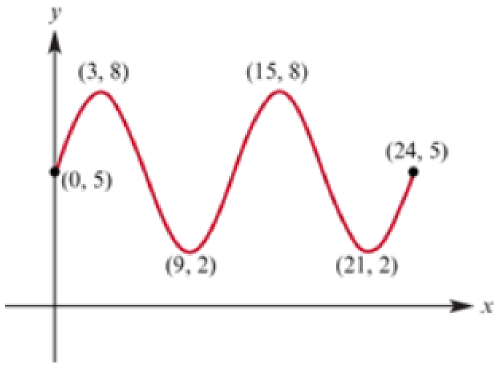
1 **a** **i** When $t = 0$ we obtain
$x = 5 + 3\sin 0 = 5 + 0 = 5.$

ii When $t = 3$ we obtain
$x = 5 + 3\sin\left(\frac{\pi}{2}\right) = 5 + 3 = 8.$

b



c **i** The maximum distance will be $5 + 3 = 8$ metres.

ii The minimum distance will be $5 - 3 = 2$ metres.

d **i** The particle will be 5 m from O when $5 + 3\sin\left(\frac{\pi}{6}t\right) = 5$

$$\begin{aligned} \sin\left(\frac{\pi}{6}t\right) &= 0 \\ \frac{\pi}{6}t &= 0, \pi, 2\pi, 3\pi, 4\pi \\ t &= 0, 6, 12, 18, 24 \end{aligned}$$

ii The particle will be 6 m from O when $5 + 3\sin\left(\frac{\pi}{6}t\right) = 6$ Since $\sin^{-1}\left(\frac{1}{3}\right) \approx 0.3398$, We know that

$$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{3}$$

$$\begin{aligned} \frac{\pi}{6}t &\approx 0.3398, \pi - 0.3398, 2\pi + 0.3398, 3\pi - 0.3398 \\ &= 0.3398, 2.8018, 6.6230, 9.0850 \\ t &\approx 0.65, 5.35, 12.65, 17.35. \end{aligned}$$

You could also simply use the solve function on your calculator.

2 $x = 4\sin\ (3t) + 4$ and $y = 2\sin\left(2t - \frac{\pi}{6}\right) + 10$

a

$$\begin{aligned} \text{When } t = 0, x &= 4\sin\ (0) + 4 \\ &= 4 \\ \text{and } y &= 2\sin\left(-\frac{\pi}{6}\right) + 10 \\ &= -1 + 10 \\ &= 9 \end{aligned}$$

b **i** The amplitude of piston A is 4.

ii The amplitude of piston B is 2.

c **i** The maximum value is $4 + 4 = 8$ and the minimum value is $4 - 4 = 0$.

ii The maximum value is $2 + 10 = 12$ and the minimum value is $10 - 2 = 8$.

d i For A, the period is $\frac{2\pi}{3}$.

ii For B, the period is π.

e The maximum value occurs when

$$\sin(3t) = 1$$

$$\text{For the first cycle, } 3t = \frac{\pi}{2}$$

$$\therefore \quad t = \frac{\pi}{6}$$

It is at the maximum distance at $\frac{\pi}{6}$ seconds .

f

$$3t = \frac{5\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \frac{13\pi}{2} \text{ or } \frac{17\pi}{2}$$

$$t = \frac{5\pi}{6} \text{ or } \frac{3\pi}{2} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6}$$

g It attains its minimum value when

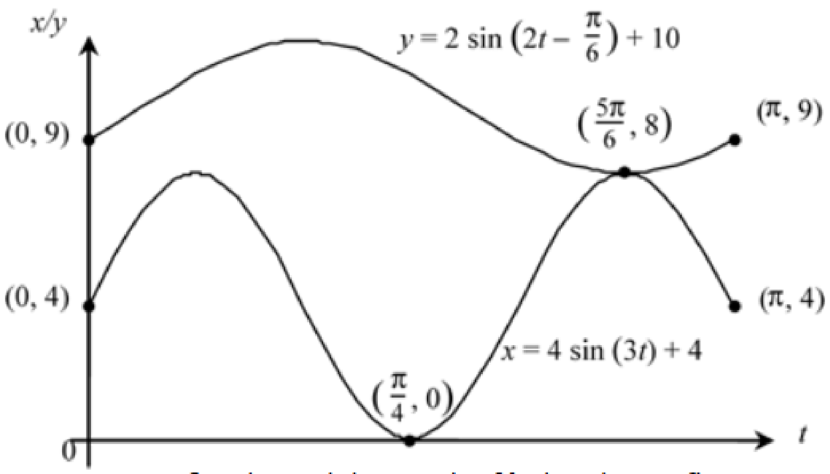
$$\sin\left(2t - \frac{\pi}{6}\right) = -1$$

$$2t - \frac{\pi}{6} = \frac{3\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \frac{11\pi}{2} \text{ or } \frac{15\pi}{2}$$

$$2t = \frac{10\pi}{6} \text{ or } \frac{22\pi}{6} \text{ or } \frac{34\pi}{6} \text{ or } \frac{46\pi}{6}$$

$$t = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{23\pi}{6}$$

h



i From parts **f** and **g** and the graph of **h** the pistons first meet when $t = \frac{5\pi}{6}$

ii They meet again when $t = \frac{17\pi}{6}$ which is 2π seconds later.

3 a The maximum value of x must be the minimum value of y.

Therefore the maximum value for x is 8 and the minimum value of y is 8.

Therefore $a + b = 8$

Also $b = 4$ if the piston goes back to position O and not beyond.

Therefore $a = 4$

The pistons meet every second. When $t = 0$, the piston A is at O.

The period of both pistons is 1 which implies $n = m = 2\pi$.

Therefore $x = 4\sin(2\pi t) + 4$

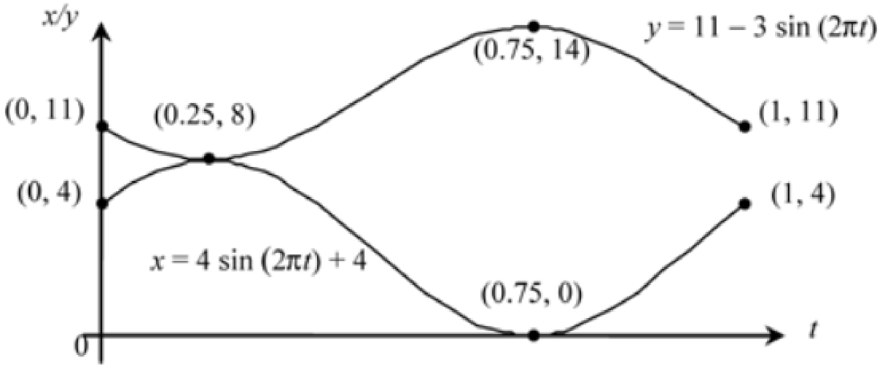
and $y = d + c\sin\ (2\pi t)$

Piston A is 8 units from O when $\sin\ (2\pi t) = 1$.

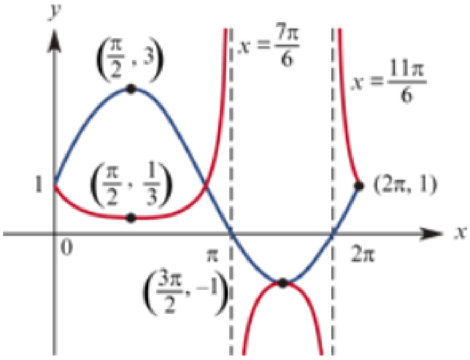
$d + c = 8$ and this is the minimum value.

Therefore one possibility is $c = -3$ and $d = 11$.

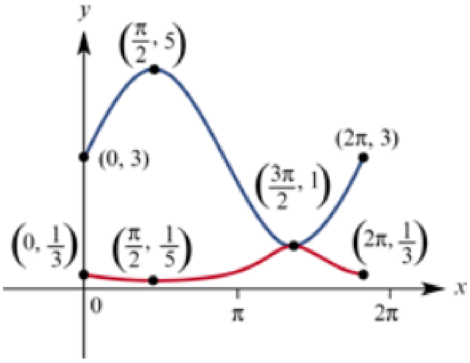
b



4 a i

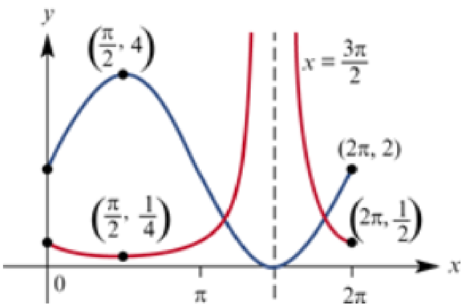


ii



b The graph will have just one vertical asymptote provided the graph of $f(x) = 2\sin x + k$ intersects the x-axis once only. This will only happen if $k = 2$.

c



5 a We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} AP &= BP \\ \sqrt{(x-1)^2 + (y-2)^2} &= \sqrt{(x-2)^2 + (y+2)^2} \\ (x-1)^2 + (y-2)^2 &= (x-2)^2 + (y+2)^2 \\ x^2 - 2x + 1 + y^2 - 4y + 4 &= x^2 - 4x + 4 + y^2 + 4y + 4 \\ y &= \frac{x}{4} - \frac{3}{8}. \end{aligned}$$

b The gradient of the line AB is

$$m = \frac{-2-2}{2-1} = -4.$$

Since $-4 \times \frac{1}{4} = -1$, the two lines are perpendicular. The midpoint of segment AB is $M(3/2, 0)$, and this is on the line $y = \frac{x}{4} - \frac{3}{8}$ since when $x = \frac{3}{2}$,

$$y = \frac{1}{4}\frac{3}{2} - \frac{3}{8} = 0.$$

c The shortest distance from the town to the road will be AM where $M(3/2, 0)$ is the midpoint of AB. This distance is

$$AM = \sqrt{\left(\frac{3}{2} - 1\right)^2 + (0-2)^2} = \frac{\sqrt{17}}{2} \text{ km}.$$

6 a There are various ways to do this question. We will find the cartesian equation corresponding to this pair of parametric equations. Solving each equation for t gives,

$$t = \frac{x+1}{4} \text{ and } t = \frac{3-y}{3}.$$

Eliminating t then gives, $\frac{3-y}{3} = \frac{x+1}{4}$ Now simply note that each of the points $(3, 0)$ and $(-1, 3)$ lie on

$$3x + 4y = 9.$$

the line since $3 \times 3 + 4 \times 0 = 9$, and

$$3 \times -1 + 4 \times 3 = 9.$$

b If we substitute $x = 4t - 1$ and $y = 3 - 3t$ into the equation for the circle we obtain,

$$\begin{aligned} (4t-1)^2 + (3-3t)^2 &= 4 \\ 16t^2 - 8t + 1 + 9 - 18t + 9t^2 &= 4 \\ 16t^2 - 8t + 1 + 9 - 18t + 9t^2 &= 4 \\ 25t^2 - 26t + 6 &= 0. \end{aligned}$$

We simply need to show that this equation has a solution. You can find the solutions, but its easier to show show that the discriminant is positive. We have,

$$\Delta = b^2 - 4ac = (-26)^2 - 4 \times 25 \times 6 > 0.$$

c We first find the cartesian equation of the line. Its gradient is

$$m = \frac{0-4}{3-(-1)} = \frac{-4}{4} = -1.$$

The equation of the line will then be

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -1(x-3) \\ y &= -x + 3. \end{aligned}$$

We can then let $x = t$ so that $y = -t + 3$.

d Substitute $x = t$ so that $y = -t + 3$ into the equation for the circle, giving,

$$\begin{aligned} t^2 + (-t+3)^2 &= 4 \\ t^2 + t^2 - 6t + 9 &= 4 \\ 2t^2 - 6t + 5 &= 0 \end{aligned}$$

We simply need to show that this equation has no solution. We look at the discriminant,

$$\Delta = b^2 - 4ac = (-6)^2 - 4 \times 2 \times 5 = -4 < 0.$$

e This can be done without using parametric equations. Simply find the equation of the line through D and B. Its gradient will be

$$m = \frac{0-k}{3-(-1)} = -\frac{k}{4}.$$

The equation of the line will then be

$$\begin{aligned} y - y_1 &= -\frac{k}{4}(x - x_1) \\ y - 0 &= -\frac{k}{4}(x-3) \\ y &= -\frac{k}{4}x + \frac{3k}{4}. \end{aligned}$$

Substituting this into the equation for the circle gives,

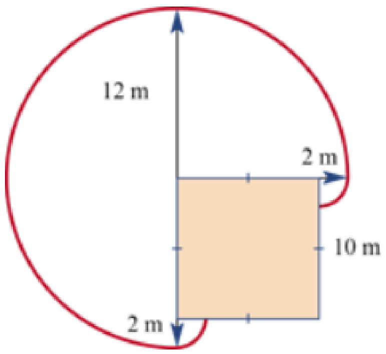
$$x^2 + (\frac{k}{4}x + \frac{3k}{4})^2 = 4.$$

Using technology or by expanding and solving by hand, this equation has solutions

$$x = \frac{-3k^2 - 4\sqrt{64-5k^2}}{k^2+16}.$$

These solutions will not exist if $64 - 5k^2 < 0$. That is, if $k > \frac{8}{\sqrt{5}}$ or $k < -\frac{8}{\sqrt{5}}$

7 a



b The area comprises a three-quarter circle of radius 12 m, and two quarter circles of radius 2 m. The total area will then be

$$\begin{aligned} A &= \frac{3}{4} \times \pi \times 12^2 + 2 \times \frac{1}{4} \times \pi \times 2^2 \\ &= 110\pi \text{ m}^2. \end{aligned}$$

c **Case 1.** If $x \leq 2$ then the area comprises

- A half circle of radius 12 ,
- a quarter circle of radius $12 - x$,
- a quarter circle of radius $12 - 10 - x = 2 - x$ and,
- a quarter circle of radius $12 - (10 - x) = 2 + x$.

The total area will then be given by the expression

$$\begin{aligned} A &= \frac{1}{2} \times \pi \times 12^2 + \frac{1}{4} \times \pi \times (12 - x)^2 + \frac{1}{4}\pi \times (2-x)^2 + \frac{1}{4}\pi \times (2+x)^2 \\ &= \frac{3\pi x^2}{4} - 6\pi x + 110\pi. \end{aligned}$$

Case 2. If $2 < x \leq 5$ then the area comprises

- A half circle of radius 12 ,
- a quarter circle of radius $12 - x$, and
- a quarter circle of radius $12 - (10 - x) = 2 + x$.

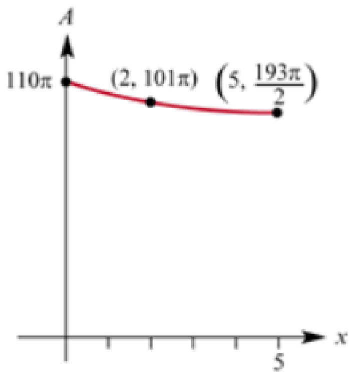
The total area will then be given by the expression

$$A = \frac{1}{2} \times \pi \times 12^2 + \frac{1}{4} \times \pi \times (12 - x)^2 + \frac{1}{4}\pi \times (2 + x)^2$$ Therefore the area is given by the hybrid function,

$$= \frac{\pi x^2}{2} - 5\pi x + 109\pi.$$

$$A(x) = \begin{cases} \frac{3\pi x^2}{4} - 6\pi x + 110\pi, & 0 \le x \le 2 \\ \frac{\pi x^2}{2} - 5\pi x + 109\pi, & 2 < x \le 5. \end{cases}$$

d

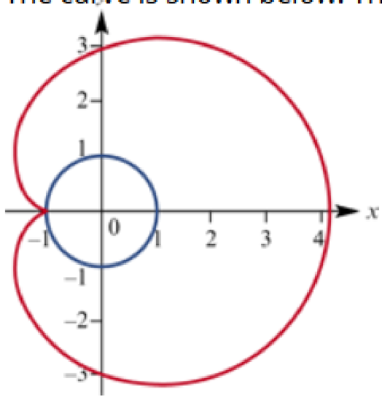


e **i** The function has a global maximum at $x = 0$, which corresponds to the corner of the shed.

ii The function has a global minimum at $x = 5$, which corresponds to the middleof one side.

8 **a** The length of the rope is π, is exactly the same as the arc length from S to the opposite side of the circle.

b The curve is shown below. The right hand side is a semicircle.



c **i** The arc length SQ will simply be $L = r\theta$

$$= 1 \times \theta$$
$$= \theta.$$

ii

$$PQ = \pi - \text{Arc}(SQ) = \pi - \theta$$

iii Since $\angle RQO = \angle SOQ = \theta$, we know that $\angle PQR = 90^\circ - \theta$. Therefore,

$$\angle RPQ = 180^\circ - (90^\circ - \theta) = \theta.$$

iv Considering right-angled triangle RPQ we have $\sin\theta = \frac{RQ}{PQ}$

$$RQ = PQ\sin\theta$$

$$= (\pi - \theta)\sin\theta.$$

v Considering right-angled triangle RPQ we have $\cos\theta = \dfrac{RP}{PQ}$

$$\begin{aligned} RP &= PQ\cos\theta \\ &= (\pi - \theta)\cos\theta. \end{aligned}$$

d First note the coordinates of Q are $Q(\cos\theta, \sin\theta)$. Therefore the x-coordinate of point P will be given by the expression

$$\begin{aligned} x &= \cos\theta - RQ \\ &= \cos\theta - (\pi - \theta)\sin\theta. \end{aligned}$$

The y-coordinate of point P will be given by the expression,

$$\begin{aligned} y &= \sin\theta + RP \\ &= \sin\theta + (\pi - \theta)\cos\theta. \end{aligned}$$

9 a i

$$\begin{aligned} \text{LHS} &= \cos 3\theta \\ &= \cos(2\theta + \theta) \\ &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta \\ &= \text{RHS} \end{aligned}$$

ii

$$\begin{aligned} \text{LHS} &= \sin 3\theta \\ &= \sin(2\theta + \theta) \\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\ &= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta \\ &= 2\sin\theta((1 - \sin^2\theta) + (1 - 2\sin^2\theta)\sin\theta \\ &= 3\sin\theta - 4\sin^3\theta \\ &= \text{RHS} \end{aligned}$$

b $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

$\cos\left(\dfrac{\pi}{3}\right) = \cos\left(3 \times \dfrac{\pi}{9}\right)$

Let $\theta = \dfrac{\pi}{9}$

Substitute in $x^3 - \dfrac{3}{4}x - \dfrac{1}{8}$.

$$\begin{aligned} x^3 - \frac{3}{4}x - \frac{1}{8} &= \cos^3\left(\frac{\pi}{9}\right) - \frac{3}{4}\cos\left(\frac{\pi}{9}\right) - \frac{1}{8} \\ &= \frac{1}{4}(4\cos^3\left(\frac{\pi}{9}\right) - 3\cos\left(\frac{\pi}{9}\right)) - \frac{1}{8} \\ &= \frac{1}{4}\cos\left(\frac{\pi}{3}\right) - -\frac{1}{8} \\ &= 0 \end{aligned}$$

d $z = \dfrac{1}{\sqrt[3]{16}}\left(\sqrt[3]{1 + \sqrt{3}i} + \sqrt[3]{1 - \sqrt{3}i}\right)$

10a **i**

$$\begin{aligned}(\text{cis }\theta)^2 &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta \\ &= \cos 2\theta + i\sin 2\theta \\ &= (\text{cis }2\theta)\end{aligned}$$

ii $(\text{cis }\theta)^3 = \text{cis }3\theta$

b

$$\begin{aligned}(\text{cis }\theta)^3 &= (\cos\theta + i\sin\theta)^3 \\ &= (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta) \\ &= (\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta)(\cos\theta + i\sin\theta) \\ &= 4\cos^3\theta - 3\cos\theta + i(3\sin\theta - 4\sin^3\theta)\end{aligned}$$

Hence $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$